The flow in compound jets

By S. C. LENNOX
King's College, Newcastle-upon-Tyne

AND D. C. PACK
Royal College of Science and Technology, Glasgow

(Received 24 May 1961 and in revised form 4 August 1962)

The Wiener–Hopf technique is applied to solve the linearized problem of a two-dimensional compound gas jet, i.e. a jet embedded in a gaseous stream of finite width. The solution is found for all combinations of supersonic and subsonic flows in jet and stream. The general nature of the solution when only one of the flows is supersonic varies according as the value of a certain quantity mk, depending upon the gas constants, Mach numbers and widths of streams, is greater than or less than unity. When $mk = 1$ the solution appears to be invalid and it is suggested that, in this critical case, a steady flow (regarded as the limit in time of an unsteady flow) may not exist. It is further shown that the solution propounded by Pai (1952) for a supersonic jet embedded in a subsonic stream is simply the asymptotic form of the general solution. The findings of Pack (1956) for a supersonic jet in a supersonic stream are confirmed and extended.

1. Introduction

A compound jet consists of a jet of gas embedded in a gaseous stream. An example is provided by the motion of the exhaust gases relative to a rocket that is itself moving through the ambient atmosphere. The gas is compressible and the speeds of the stream and jet may be either subsonic or supersonic. Here it will be assumed that the pressure difference between the jet and stream is small, so that the fluctuations in the jet boundary are small. The method of small perturbations is used to linearize the problem.

Much attention has been given to the problem of a subsonic or supersonic jet that emerges into a medium at rest (Rayleigh 1916; Chaplygin 1904; Jacob 1936; Mach & Salcher 1890; Emden 1899; Prandtl 1904, 1907; Pack 1948, 1950) but the problem of a supersonic jet that emerges into a supersonic stream of infinite width has only recently been studied (Pai 1952; Kawamura 1952; Pack 1956; Ehlers & Strand 1958). Pai obtained a solution by the method of characteristics; Pack found a more general form of solution by employing the Laplace Transform and was able to examine the wave structure in the jet and the stream and the fluctuations in the jet boundary. The work of both of these authors was further generalized by Ehlers & Strand who discussed the motion of a jet inclined at a small angle to the main stream. The problem of a supersonic jet emerging into a subsonic stream of infinite width was considered by Pai (1952) but he did not fully specify the boundary conditions, no account being taken of the fact that the disturbances would spread upstream in the subsonic region. This omission was noted by Klunker & Harder (1952) who indicated that the solution as given by Pai could be valid only far downstream of the jet orifice.

When the supersonic jet is made to emerge into a subsonic stream of finite width it is found that the problem is amenable to treatment by the Wiener–Hopf technique (Paley & Wiener 1934; Noble 1958) and that the results obtained by Pai are simply the asymptotic form of the general solution. By a simple modification of the notation the discussion is extended to compound jets involving other combinations of subsonic and supersonic speeds.

2. Formulation of the problem

Let an ideal gas pass through a straight-walled nozzle $-h < y < h$, $-\infty < x < 0$ and emerge as a two-dimensional jet in the region $-h < y < h$, $0 < x < +\infty$ into a stream of ideal gas which occupies the region $h < |y| < H$, $-\infty < x < +\infty$ and flows in the same direction as the jet. The stream is bounded by rigid walls at $y = \pm H$ and the flow regions are as shown in figure 1.

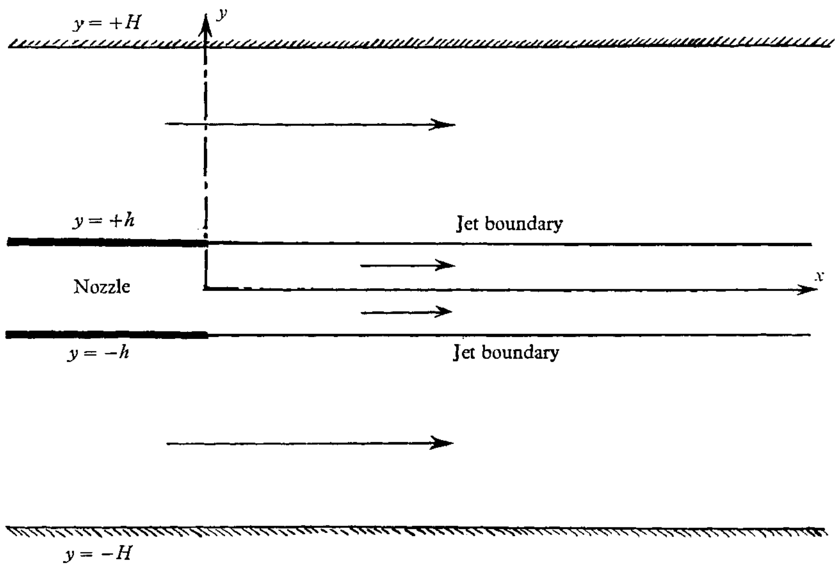

FIGURE 1. Schematic diagram of flow regions.

In the undisturbed state the jet is taken to be a uniform parallel stream of inviscid, non-heat-conducting gas with velocity U_1, Mach number M_1, density ρ_1 and pressure p_1. The outer stream is supposed to consist of an inviscid non-heat-conducting gas with undisturbed velocity U_2, Mach number M_2, density ρ_2 and pressure p_2. The pressure difference $(p_1 - p_2)$ is supposed small compared with p_1, p_2 or $\rho_2 U_2^2$, the variation of the streams from their original parallel flow is assumed to be small and linearized theory is used.

The velocity potentials in the inner and outer streams, respectively, may be written

$$\Phi_1 = U_1(x+\phi_1), \quad \Phi_2 = U_2(x+\phi_2),$$

where $\phi_1(x,y)$ and $\phi_2(x,y)$ are called the perturbation potentials. The boundary conditions are formed by ensuring that:

(i) the direction of flow is continuous across the jet boundary,
(ii) there is no flow over either the rigid walls or the axis of symmetry, and
(iii) the pressure is continuous across the jet boundary.

In order to apply the Wiener–Hopf technique it is convenient to write

$$\phi(x,y) = \phi_+(x,y)+\phi_-(x,y),$$

where the functions ϕ_+ and ϕ_- are defined by

$$\phi_+(x,y) = \begin{cases}\phi(x,y) & \text{if} \quad x>0,\\ 0 & \text{if} \quad x<0\end{cases} \quad \text{and} \quad \phi_-(x,y) = \begin{cases}0 & \text{if} \quad x>0,\\ \phi(x,y) & \text{if} \quad x<0.\end{cases}$$

The boundary conditions may then be written

$$\frac{\partial\phi_+(x,h+0)}{\partial y} = \frac{\partial\phi_+(x,h-0)}{\partial y}, \quad \frac{\partial\phi_-(x,h+0)}{\partial y} = \frac{\partial\phi_-(x,h-0)}{\partial y} = 0, \qquad (1, 2)$$

$$\frac{\partial\phi(x,0)}{\partial y} = 0, \quad \frac{\partial\phi(x,H-0)}{\partial y} = 0 \qquad (3, 4)$$

and

$$\frac{\partial\phi_+(x,h+0)}{\partial x} - l^2\frac{\partial\phi_+(x,h-0)}{\partial x} = -\epsilon, \qquad (5)$$

where $l^2 = \rho_1 U_1^2/\rho_2 U_2^2$ for general gases,
$= \gamma_1 M_1^2/\gamma_2 M_2^2$, for polytropic gases with constant ratios of specific heats, γ_1 and γ_2, respectively,

and $\epsilon = (p_1-p_2)/\rho_2 U_2^2$.

The boundary condition (5), which implies the continuity in the pressure over the jet boundary, is in the form used by Pack (1956). The perturbation potentials ϕ_1 and ϕ_2 satisfy the differential equations

$$\frac{\partial^2\phi_i(x,y)}{\partial y^2} - (M_i^2-1)\frac{\partial^2\phi_i}{\partial x^2} = 0 \quad (i = 1, 2). \qquad (6)$$

Denote the Fourier transform of a function by means of a bar, and in particular put

$$\overline{\phi}(\lambda,y) = \overline{\phi}_+(\lambda,y)+\overline{\phi}_-(\lambda,y) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty} \exp(i\lambda x)\,\phi(x,y)\,dx.$$

On taking the transforms of the equations (1) to (6) the problem is reduced to finding the function $\overline{\phi}$ that satisfies the differential equations

$$d^2\overline{\phi}(\lambda,y)/dy^2 + \lambda^2(M_i^2-1)\,\overline{\phi}(\lambda,y) = 0 \quad (i = 1, 2), \qquad (7)$$

and is subject to the boundary conditions

$$\frac{d\overline{\phi}_+(\lambda,h+0)}{dy} = \frac{d\overline{\phi}_+(\lambda,h-0)}{dy}, \quad \frac{d\overline{\phi}_-(\lambda,h+0)}{dy} = \frac{d\overline{\phi}_-(\lambda,h-0)}{dy} = 0, \qquad (8, 9)$$

$$\frac{d\overline{\phi}(\lambda,0)}{dy} = 0, \quad \frac{d\overline{\phi}(\lambda,H-0)}{dy} = 0 \qquad (10, 11)$$

and

$$\overline{\phi}_+(\lambda,h+0) - l^2\overline{\phi}_+(\lambda,h-0) = \epsilon/\lambda^2\sqrt{2\pi}. \qquad (12)$$

In deriving (12) from (5) the perturbation potentials ϕ_1 and ϕ_2 have been chosen so that $\phi_+(0,h+0)$ and $\phi_+(0,h-0)$ are both zero. It is necessary to examine the convergence of the various Fourier integrals and this is undertaken in the discussion of the problems.

There are four possible combinations of flow depending on whether M_1 and M_2 are greater than or less than unity. These are considered in the following sections. Since $y = 0$ is a line of symmetry the solutions will be written down for the upper half-plane only.

3. The supersonic jet in a subsonic stream ($M_1 > 1, M_2 < 1$)

Write $M_1^2 - 1 = \beta_1^2$ and $1 - M_2^2 = \beta_2^2$, then it follows that the potential which satisfies (7) and the boundary conditions (10) and (11) may be written in the form

$$\bar{\phi}(\lambda, y) = \bar{\phi}_+(\lambda, y) + \bar{\phi}_-(\lambda, y) = \begin{cases} A(\lambda)\cos\lambda\beta_1 y & \text{for} \quad 0 \leqslant y \leqslant h-0, \\ B(\lambda)\cosh\lambda\beta_2(H-y) & \text{for} \quad h+0 \leqslant y \leqslant H-0, \end{cases} \tag{13}$$

where $A(\lambda)$ and $B(\lambda)$ are functions of λ only. The conditions (8) and (9) may now be employed to determine $A(\lambda)$ and $B(\lambda)$ in terms of $d\bar{\phi}_+(\lambda, h \pm 0)/dy$ and then satisfaction of condition (12) leads to

$$\bar{v}_+(\lambda, h)\, K(\lambda)/\beta_2 = -\epsilon/\lambda(2\pi)^{\frac{1}{2}} - G_-(\lambda, h), \tag{14}$$

where

$$K(\lambda) = \coth\lambda\beta_2 L - m\cot\lambda\beta_1 h, \tag{15}$$

$$m = l^2\beta_2/\beta_1, \quad L = H - h, \tag{16, 17}$$

$$\bar{v}_+(\lambda, h) = d\bar{\phi}_+(\lambda, h \pm 0)/dy \tag{18}$$

and

$$G_-(\lambda, h) = \lambda[\bar{\phi}_-(\lambda, h+0) - l^2\bar{\phi}_-(\lambda, h-0)]. \tag{19}$$

Equation (14) is of 'Wiener–Hopf form' (Noble 1958) and it is necessary to determine whether the terms can be rearranged in such a manner that one side of the equation is analytic in an upper half of the complex λ-plane, whilst the other side is analytic in an overlapping lower half-plane, both sides being analytic in a common strip. This rearrangement may be performed after investigating the regions of analyticity of the functions $\bar{v}_+(\lambda, h)$, $K(\lambda)$ and $G_-(\lambda, h)$.

The transform $\bar{v}_+(\lambda, h)$ is given by

$$\bar{v}_+(\lambda, h) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \exp i\lambda x \frac{\partial\phi_+(x, h)}{\partial y}\, dx.$$

The perturbation velocity, and hence $\partial\phi_+(x, h)/dy$ must be bounded as $x \to +\infty$ and it follows (Noble 1958, p. 23) that $\bar{v}_+(\lambda, h)$ exists and is analytic in the region $\operatorname{Im}\lambda > 0$. The function $G_-(\lambda, h)$ is given by

$$G_-(\lambda, h)/i = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^0 \exp i\lambda x \frac{\partial\phi_-(x, h+0)}{\partial x}\, dx - \frac{l^2}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^0 \exp i\lambda x \frac{\partial\phi_-(x, h-0)}{\partial x}\, dx$$

but, due to the nature of supersonic flow, $\partial\phi_-(x, h-0)/\partial x$ is zero and so the second integral vanishes. Further, in the region $h+0 < y < H-0$, as $x \to -\infty$, $\phi_-(x, y)$ is assumed to be asymptotic to $\Sigma\, A_n \exp[n\pi x/\beta_2 L]\cos[n\pi(H-y)/L]$. This is tantamount to the statement that the asymptotic form is the bounded solution of (7) ($i = 2$) which would be obtained, as $x \to -\infty$, if the rigid walls at $y = h$ extended to positive infinity. The function $G_-(\lambda, h)$ therefore exists and defines a function of λ analytic in the half-plane $\operatorname{Im}\lambda < \pi/\beta_2 L$. The function $1/\lambda$ is analytic in the region $\operatorname{Im}\lambda > 0$ and there is a simple pole at $\lambda = 0$. Finally, there remains the discussion of the function $K(\lambda)$. It is evident that it has poles on both the real and imaginary axes, the latter situated at the zeros of $\tanh \lambda\beta_2 L = 0$. The function $K(\lambda)$ is certainly analytic in the region

$$0 < \operatorname{Im}\lambda < \pi/\beta_2 L.$$

Suppose now that $K(\lambda)$ can be factorized and written in the form

$$K(\lambda) = K_+(\lambda)/K_-(\lambda)$$

where $K_+(\lambda)$ is analytic and non-zero in the upper half-plane $\operatorname{Im}\lambda > 0$ and $K_-(\lambda)$ is analytic and non-zero in the lower half-plane $\operatorname{Im}\lambda < \delta < \pi/\beta_2 L$ where δ is a positive real constant to be determined (see after equation (28)). With this supposition the equation (14) may be rewritten

$$\bar{v}_+(\lambda, h)\, K_+(\lambda)/\beta_2 + \epsilon K_-(0)/\lambda(2\pi)^{\frac{1}{2}} = -\epsilon[K_-(\lambda) - K_-(0)]/\lambda(2\pi)^{\frac{1}{2}} - G_-(\lambda, h)\, K_-(\lambda). \tag{20}$$

The left side of (20) is analytic in the upper half-plane $\operatorname{Im}\lambda > 0$, the right side is analytic in the lower half-plane $\operatorname{Im}\lambda < \delta$ and both are analytic in the common strip. It follows that one side of (20) is the analytical continuation of the other; both may therefore be represented by an integral function $E(\lambda)$. By an investigation of the growth order of each side as $|\lambda| \to \infty$ in the appropriate half-plane, it may be shown that $E(\lambda)$ is identically zero. Thus

$$\bar{v}_+(\lambda, h) = -\epsilon\beta_2 K_-(0)/\lambda(2\pi)^{\frac{1}{2}} K_+(\lambda) \tag{21}$$

and

$$G_-(\lambda, h) = -\epsilon[K_-(\lambda) - K_-(0)]/\lambda(2\pi)^{\frac{1}{2}} K_+(\lambda). \tag{22}$$

The equation (21), or (22), may be used to find the perturbation potentials in the entire flow regions. In addition to the perturbation potentials, the displacement of the jet boundary may be examined by writing the equation of the boundary in the form

$$y = h[1 + f(x)], \quad f(x) = 0(\epsilon) \quad (0 < x < \infty),$$

from which it follows that

$$hf'(x) = \partial\phi_+(x, h)/\partial y$$

to the approximation of the linearized theory. The Fourier-Transform theorem may be applied to give $\bar{f}(\lambda)$ in the form $-i\lambda\bar{f}(\lambda) = \bar{v}_+(\lambda, h)$.

It is noted that $\bar{f}(\lambda)$ is analytic in the upper half-plane $\operatorname{Im}\lambda > 0$. The working is simplified by a change of variable and to this end the following transformations are introduced:

$$\left.\begin{aligned} \xi &= \lambda\beta_1 h, & k &= \beta_2 L/\beta_1 h, \\ t &= x/\beta_1 h, & v_+(x, y) &= \epsilon\beta_2 V_+(t, y)/(2\pi)^{\frac{1}{2}}, \\ f(x) &= i\epsilon\beta_1\beta_2 F(t)/(2\pi)^{\frac{1}{2}}, & \phi(x, y) &= \epsilon h\beta_2\psi(t, y)/(2\pi)^{\frac{1}{2}}. \end{aligned}\right\} \tag{23}$$

It follows that

$$\bar{V}_+(\xi, h) = -Q_-(0)/\xi Q_+(\xi) \tag{24}$$

and

$$\bar{F}(\xi) = -Q_-(0)/\xi^2 Q_+(\xi), \tag{25}$$

where

$$Q(\xi) = K(\lambda) = Q_+(\xi)/Q_-(\xi) = \coth k\xi - m \cot \xi, \tag{26}$$

and the strip in which $Q(\xi)$ is analytic is given by $0 < \text{Im}\,\xi < \delta'$ where $\delta'(=\delta\beta_1 h)$ is a real positive constant.

It is now necessary to investigate the factorization of $Q(\xi)$. From

$$Q(\xi) = Q_+(\xi)/Q_-(\xi) = [\tan\xi - m\tanh k\xi]/\tan\xi\tanh k\xi$$

and the fact (Hardy 1938, p. 480) that $\tan\xi - m\tanh k\xi$ has an infinity of real and imaginary zeros but no complex ones, while the zeros of $\tan\xi$ and of $\tanh k\xi$ are well known, $Q(\xi)$ may be expanded in an infinite product by use of the Weierstrassian factor theorem. The expansion is different according as mk is greater than, less than or equal to unity. Write $\xi = \sigma + i\tau$. When $mk > 1$ the roots of $\tan\xi - m\tanh k\xi$ are given by $\xi = 0$, $\xi = \pm\sigma_0$, $\xi = \pm\sigma_n$ and $\xi = \pm i\tau_n$ where $0 < \sigma_0 < \frac{1}{2}\pi$, $n\pi < \sigma_n < \frac{1}{2}(2n+1)\pi$, $n\pi < k\tau_n < \frac{1}{2}(2n+1)\pi$ and $n = 1, 2, 3, \ldots$. When $mk < 1$ the roots are given by $\xi = 0$, $\xi = \pm\sigma_n$, $\xi = \pm i\tau_0$ and $\xi = \pm i\tau_n$ where $n\pi < \sigma_n < \frac{1}{2}(2n+1)\pi$, $0 < k\tau_0 < \frac{1}{2}\pi$, and $n\pi < k\tau_n < \frac{1}{2}(2n+1)\pi$. When $mk = 1$ the function $\tan\xi - m\tanh k\xi$ has a triple zero at $\xi = 0$ and the other roots are given by $\xi = \pm\sigma_n$ and $\xi = \pm i\tau_n$ with $n = 1, 2, 3, \ldots$. It may be noted that for large values of n, $\sigma_n \sim n\pi + \theta$ and $k\tau_n \sim n\pi + \omega$ where $\theta = \tan^{-1}(m)$ and $\omega = \tan^{-1}(1/m)$. The growth orders of (suitably chosen) functions $Q_\pm(\xi)$ as $|\xi| \to \infty$ are required in the subsequent analysis and these may be shown by an analysis similar to that of Noble (1958, p. 128) to be

$$Q_\pm(\xi) \sim O(\xi^q) \quad \text{for} \quad mk \geq 1 \quad \text{and} \quad Q_\pm(\xi) \sim O(\xi^{q-1}) \quad \text{for} \quad mk < 1,$$

where $0 < q < \frac{1}{2}$.

3.1. *The case mk > 1*

When $mk > 1$

$$Q(\xi) = \frac{(1-mk)(1-\xi^2/\sigma_0^2)}{k\xi}\prod_1^\infty \frac{(1+\xi^2/\tau_n^2)(1-\xi^2/\sigma_n^2)}{(1+k^2\xi^2/n^2\pi^2)(1-\xi^2/n^2\pi^2)}.$$

Factorization gives

$$Q_+(\xi) = \frac{(1-mk)(1-\xi^2/\sigma_0^2)}{k\xi}\prod_1^\infty \frac{[(1-i\xi/\tau_n)\exp(ik\xi/n\pi)][1-\xi^2/\sigma_n^2]}{[(1-ik\xi/n\pi)\exp(ik\xi/n\pi)][1-\xi^2/n^2\pi^2]} \quad (27)$$

and

$$Q_-(\xi) = \prod_1^\infty \frac{(1+ik\xi/n\pi)\exp(-ik\xi/n\pi)}{(1+i\xi/\tau_n)\exp(-ik\xi/n\pi)}. \quad (28)$$

The exponential terms in (27) and (28) are necessary to ensure the absolute convergence of the product functions. The choice of terms is not unique but the present choice ensures that $Q_\pm(\xi)$ have algebraic growth as $|\xi| \to \infty$. The factorization of $Q(\xi)$ has been formed in such a way that $Q_+(\xi)$ is analytic and non-zero in the upper half-plane $\text{Im}\,\xi > 0$, while $Q_-(\xi)$ is analytic and non-zero in the lower half-plane $\text{Im}\,\xi < \pi/k$ (this determines the value of the constants δ' and δ), and further, that $Q_-(0) = 1$.

From (23), (13), (8), (9) and(18) it may be shown that

$$\bar{\Psi}(\xi, y) = -\cos(\xi y/h)\,\bar{V}_+(\xi, h)/\xi\sin\xi$$

for the region $0 \leq y \leq h-0$ and

$$\bar{\Psi}(\xi, y) = -(\beta_1/\beta_2)\cosh[k\xi(H-y)/L]\,\bar{V}_+(\xi, h)/\xi\sinh k\xi$$

for the region $h+0 \leqslant y \leqslant H-0$. The inverse transform theorem yields

$$\psi(t,y) = -\frac{1}{(2\pi)^{\frac{1}{2}}}\int_C \frac{\exp(-i\xi t)\cos(\xi y/h)\,\overline{V}_+(\xi,h)}{\xi\sin\xi}\,d\xi \quad (0 \leqslant y \leqslant h-0), \tag{29}$$

and

$$\psi(t,y) = -\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{\beta_1}{\beta_2}\int_C \frac{\exp(-i\xi t)\cosh[k\xi(H-y)/L]\,\overline{V}_+(\xi,h)}{\xi\sinh k\xi}\,d\xi \quad (h+0 \leqslant y \leqslant H-0), \tag{30}$$

where C is a path drawn from $-\infty$ to $+\infty$ in the strip of analyticity. The path C will, as required, be closed by an infinite semicircle in either the upper or lower half-planes, drawn so as to pass between the poles of the integrands on both axes; the integrals (29) and (30) will then be evaluated by using Cauchy's residue theorem.

In (29) let the contour be closed by a semicircle in the upper-half-plane; the integral taken round this semicircle vanishes provided that $t-1+y/h<0$. Since the integrand (29) has no poles in the upper half-plane it follows, in terms of the original variables, that $\phi(x,y)$ is zero in the region $y-h+x/\beta_1<0$, $0 \leqslant y \leqslant h-0$; this shows that there is no disturbance in the supersonic jet upstream of the leading characteristic from the jet orifice, a result to be expected on physical grounds. Downstream of this characteristic there is a disturbance given by evaluating the residues at the poles of the integrand of (29) in the lower half-plane. In (30) there are non-zero residues whether the contour be closed in the upper or the lower half-plane. This indicates again, as expected, that disturbances spread both up and down stream in the subsonic flow.

The residues may now be evaluated and the following results obtained.

For the jet,

$$\begin{aligned}\frac{\phi(x,y)}{\epsilon h\beta_2} = {} & \frac{kx}{\beta_1 h(mk-1)} + \frac{Ck}{mk-1} \\ & -\sum_1^\infty \frac{\exp(-\tau_r x/\beta_1 h)\cosh(\tau_r y/h)\,[Q_-(-i\tau_r)]^{-1}}{\tau_r^2\sinh\tau_r[k\operatorname{cosec}^2 k\tau_r - m\operatorname{cosech}^2\tau_r]} \\ & +\sum_0^\infty \frac{\cos(\sigma_r y/h)\,[A_r\sin(x\sigma_r/\beta_1 h)+B_r\cos(x\sigma_r/\beta_1 h)]}{\sigma_r^2\sin\sigma_r\,[k\operatorname{cosech}^2 k\sigma_r - m\operatorname{cosec}^2\sigma_r]}\end{aligned} \tag{31}$$

in the region $0 \leqslant y \leqslant h-0$, $y-h+x/\beta_1>0$ and

$$\phi(x,y)/\epsilon h\beta_2 = 0 \text{ in the region } 0 \leqslant y \leqslant h-0, y-h+x/\beta_1<0. \tag{32}$$

For the stream,

$$\begin{aligned}\frac{\phi(x,y)}{\epsilon h\beta_1} = {} & \frac{x}{h\beta_1(mk-1)} + \frac{C}{mk-1} \\ & -\sum_1^\infty \frac{\exp(-\tau_r x/\beta_1 h)\cos[\tau_r k(H-y)/L]\,[Q_-(-i\tau_r)]^{-1}}{\tau_r^2\sin k\tau_r\,[k\operatorname{cosec}^2 k\tau_r - m\operatorname{cosech}^2\tau_r]} \\ & +\sum_0^\infty \frac{\cosh[\sigma_r k(H-y)/L]\,[A_r\sin(x\sigma_r/\beta_1 h)+B_r\cos(x\sigma_r/\beta_1 h)]}{\sigma_r^2\sinh k\sigma_r\,[k\operatorname{cosech}^2 k\sigma_r - m\operatorname{cosec}^2\sigma_r]}\end{aligned} \tag{33}$$

in the region $h+0 \leqslant y \leqslant H-0$, $x>0$ and

$$\frac{\phi(x,y)}{\epsilon h\beta_1} = \sum_1^\infty \frac{\exp(r\pi x/\beta_2 L)\cos[r\pi(H-y)/L]\,\overline{V}_+(ir\pi/k,h)}{r\pi(-1)^{r+1}} \tag{34}$$

in the region $h+0 \leqslant y \leqslant H-0$, $x < 0$, where

$$C = -iQ'_{-}(0) \quad \text{and} \quad A_r + iB_r = 2[Q_{-}(-\sigma_r)]^{-1}.$$

Downstream of the jet exit, the perturbation potentials fall into three parts. The first simply indicates a steady component of the perturbation velocity parallel to the jet axis, the second contains the terms which decay exponentially with the distance from the exit and the third part consists of fluctuating terms. These fluctuations are not periodic, in general, due to the nature of the roots σ_r, but when $k \to \infty$ the terms are almost periodic. Upstream of the jet orifice the potential is zero in the supersonic region and decays exponentially in the subsonic region. Comparison with the result obtained by Pai (1952) may be made by letting $x \to \infty$ in (31) and (33) and then considering the situation as $k \to \infty$. The roots σ_r rapidly approximate to those of $\tan\theta = m$ so that for the supersonic jet in a subsonic stream of infinite width the equation (31) gives, in the limit,

$$\frac{\phi(x,y)}{\epsilon h \beta_2} \sim \frac{x}{m\beta_1 h} + \sum_0^\infty \cos[(\theta + r\pi)\, y]\,[A_{1r}\cos[(\theta + r\pi)\, y/\beta_1 h] + B_{1r}\sin[(\theta + r\pi)\, y/\beta_1 h]] \tag{35}$$

where $0 < \theta < \frac{1}{2}\pi$ and A_{1r}, B_{1r} are constants dependent on A_r and B_r. The equation (35) is identical in form with that given by Pai except for the first term which does not appear in his solution. This is, however, simply a difference in the notation, the U_1 of the present paper being the undisturbed jet velocity, whereas the equivalent velocity in Pai's analysis is taken to be the mean velocity in the jet when the pressure is equal to that of the undisturbed stream. Similarly, by considering equation (33) it may be shown that

$$\phi(x,y)/\epsilon h \beta_1 \sim \sum_0^\infty \exp[-\beta_2(r\pi + \theta)\, y/\beta_1 h] \times [A_{2r}\cos[(r\pi + \theta)\, y/\beta_1 h] + B_{2r}\sin[(r\pi + \theta)\, y/\beta_1 h]] \tag{36}$$

as x, $k \to \infty$, where A_{2r} and B_{2r} are constants dependent on A_r and B_r. This is again identical with the solution given by Pai.

The behaviour of the jet boundary may be examined by use of the displacement function $f(x)$. The initial slope of the jet boundary and the ultimate jet width are of some interest. The theory of residues leads to the result

$$f(x) \sim \epsilon\beta_1\beta_2 k/(mk-1) + \text{terms in } {\sin \atop \cos}\, (\sigma_r x/\beta_1 h).$$

This shows that, far downstream, the width of the boundary fluctuates about a mean value $h[1 + \epsilon\beta_1\beta_2 k/(mk-1)]$, indicating, for example, that when there is over-pressure in the jet ($\epsilon > 0$) there is a small overall expansion in the jet width. This mean value may be obtained very simply by using the equations of conservation of mass and energy between upstream and downstream infinity and the condition for continuity of pressure across the jet boundary. It is easy to show that when, for example, there is an ultimate expansion of the jet, the final pressure lies below both p_1 and p_2, as would be expected from a consideration of the behaviour of streamtubes in the supersonic jet and subsonic stream.

An estimate of $f(x)$ near the jet orifice, found by taking the inverse transform of the asymptotic form of $\bar{F}(\xi)$ of (25), shows that

$$f(x)/\epsilon \sim O(x^{q+1}) \quad \text{as} \quad x \to 0+,$$

where $0 < q < \frac{1}{2}$. Thus, initially, the jet expands (when $\epsilon > 0$) and the boundary curve lies between the limiting curves $f(x)/\epsilon = O(x)$ and $f(x)/\epsilon = O(x^{\frac{3}{2}})$.

3.2. *The case $mk < 1$*

As has been observed, the pair of real roots

$$\xi = \pm\sigma_0, \quad 0 < \sigma_0 < \tfrac{1}{2}\pi \quad \text{of} \quad \tan\xi - m\tanh k\xi = 0$$

are now replaced by a pair of imaginary roots $\xi = \pm i\tau_0$, $0 < k\tau_0 < \frac{1}{2}\pi$. This has the effect of modifying the functions $Q_+(\xi)$ and $Q_-(\xi)$, altering their orders at infinity and so leading to a different behaviour of the jet boundary.

The strip of analyticity in the ξ-plane is given by $0 < \operatorname{Im}\xi < \delta'(=\tau_0)$. The evaluation of the contour integrals proceeds as in the previous case and the potential functions are obtained in a form differing only in the number of terms in the summations. The displacement function far downstream of the jet exit is given by

$$f(x) \sim -\epsilon\beta_1\beta_2 k/(1-mk) + \text{ terms in } \begin{matrix}\sin\\ \cos\end{matrix}(\sigma_r x/\beta_1 h),$$

showing that the boundary fluctuates about a mean value $h[1-\epsilon\beta_1\beta_2 k/(1-mk)]$ and that in the case of over-pressure there is an overall decrease in the mean jet width, a result opposite to that obtained for the case $mk > 1$.

The behaviour of the boundary at the orifice is again found by an examination of the asymptotic form of $\bar{F}(\xi)$. This leads to

$$f(x)/\epsilon \sim O(x^q) \quad (0 < q < \tfrac{1}{2}),$$

and it is seen that the initial shape lies between the curves $f(x)/\epsilon \sim O(1)$ and $f(x)/\epsilon \sim O(x^{\frac{1}{2}})$. The initial slope is $O(x^{-\alpha})$, where $\frac{1}{2} < \alpha < 1$, and is infinite. An infinity of this kind while physically anomalous occurs in the solution of other plane subsonic flow problems and may be interpreted as due to an over-specification of boundary conditions. Woods (1961, p. 261) gives the example of subsonic flow from a nozzle into a medium at rest and indicates that the anomalous situation arising at the exit may be avoided by a proper shaping of the nozzle or by the assumption of separating flow. Similar considerations would appear to apply here. The physical possibility of maintaining the pressure differences in the compound jet would depend on an adjustment of the boundary, failing which one might expect the annulment of these differences by upstream propagation of disturbances through the subsonic region.

3.3. *The case $mk = 1$*

This case is characterized by the behaviour of the function $\tan\xi - m\tanh k\xi$ at $\xi = 0$. The function vanishes there like ξ^3. Thus, in particular, $Q_+(\xi)$ has a simple zero at the origin (whereas in the previous two cases it had a simple pole).

Consider the displacement function. Near $\xi = 0$ the function $\bar{F}(\xi)$ of (25) has the expansion $3k[1+O(\xi)]/(1+k^2)\,\xi^3$ and it is seen that as $x \to \infty$

$$f(x) \sim a_1x^2+b_1x+c_1+\ldots,$$

where a_1, b_1, c_1, are constants. In fact, the jet leaves the exit with a zero gradient, undergoes an initial expansion but continues to diverge. The solution is physically impossible but bears a strong resemblance to well-known resonance phenomena.

Comparison may be drawn between this problem in gas dynamics and a problem discussed by Stoker (1957) of waves created by a disturbance on the surface of a running stream of finite depth h. When the stream flows with an undisturbed velocity U, Stoker shows that the boundary condition on the free surface leads to a discussion of the zeros of $h\xi-(gh/U^2)\tanh h\xi$. This function corresponds, in so far as the two problems may be compared, with $\tan\xi - m\tanh k\xi$, where both functions vanish like ξ^3 at $\xi = 0$ in the critical cases $(gh/U^2) = 1$ and $mk = 1$, respectively. Stoker shows that, in the critical case, the wave amplitude and perturbation velocities all tend to infinity with the distance downstream. However, by a more detailed examination of the general unsteady problem he is able to discuss the behaviour of the transient terms and to show that they are $O(t^{\frac{1}{3}})$ as $t \to \infty$ and that consequently the steady state is never attained. The conclusion is reached that it is no longer possible to apply the assumptions of the linearized theory in this case. It seems likely that if the linearized unsteady flow in the compound jet considered here could be analysed in a similar way the transient terms would increase with time. The further discussion of the solution in the critical case must therefore await the solution of the linearized unsteady problem, or an attack on the full non-linear equations of steady flow.

The problem of a subsonic jet in a supersonic stream ($M_1 < 1$, $M_2 > 1$) is mathematically identical with that discussed above. It is only necessary to interchange the indices 1 and 2, invert m and k, and put $H-y$ for y in the formulation of the problem. The disturbances spread in all directions inside the jet but only downstream of the leading characteristics in the outer stream.

4. The supersonic jet in a supersonic stream ($M_1 > 1, M_2 > 1$)

Although a solution to this problem may be obtained without employing the Wiener-Hopf technique (Pai 1952; Pack 1956) it is, nevertheless, of interest to note that by a small modification of §3 the solution may be obtained at once.

With $M_1^2-1 = \beta_1^2$ and $M_2^2-1 = \beta_2^2$ and all other notation as in the previous section, the following expressions may be obtained:

$$\bar{V}_+(\xi,h) = 1/\xi Q(\xi), \tag{37}$$

$$Q(\xi) = \cot k\xi + m\cot\xi. \tag{38}$$

It may be shown (Carslaw & Jaeger 1959, p. 324) that $Q(\xi)$ has all of its poles and zeros real and simple, so that $Q(\xi)$ is analytic and non-zero in the upper half-plane $\operatorname{Im}\xi > 0$. Since $\bar{V}_+(\xi,h)$ and $1/\xi$ are both analytic in this region no factorization is necessary and (37) takes the place of (24). Similarly

$$\bar{F}(\xi) = 1/\xi^2 Q(\xi) \tag{39}$$

replaces the equation (25).

The perturbation potentials and the displacement function may be obtained as in § 3. For example, in the jet

$$\frac{\phi(x,y)}{\epsilon h\beta_2} = \frac{xk}{[\beta_1 h(mk+1)]} - 2\sum_1^\infty \frac{\cos(\sigma_r y/h)\sin(\sigma_r x/\beta_1 h)}{\sigma_r^2 \sin\sigma_r [k \operatorname{cosec}^2 k\sigma_r + m\operatorname{cosec}^2\sigma_r]}$$

for the region $y-h+x/\beta_1 > 0$, $0 \leqslant y \leqslant h-0$ and $\phi(x,y)/\epsilon h\beta_2 = 0$ for the region $y-h+x/\beta_1 < 0$, $0 \leqslant y \leqslant h-0$, σ_r being the positive non-zero roots of $\tan\xi + m\tan k\xi = 0$. The displacement function is given by

$$\frac{f(x)}{\epsilon\beta_1\beta_2} = \frac{k}{(mk+1)} - 2\sum_1^\infty \frac{\cos(\sigma_r x/\beta_1 h)}{\sigma_r^2[k\operatorname{cosec}^2 k\sigma_r + m\operatorname{cosec}^2\sigma_r]}. \tag{40}$$

It may be noted that in the over-pressure case ($\epsilon > 0$) the jet boundary expands to a mean width given by $y = h[1+\epsilon\beta_1\beta_2 k/(mk+1)]$ with quasi-periodic fluctuations about the mean. If the outer stream wall be made infinitely wide ($k \to \infty$) then the ultimate mean width is $h[1+\epsilon\beta_1\beta_2/m]$ which is just the value found by Pack (1956). The extra terms in (40) are due entirely to the interference arising from the waves reflected from the outer walls.

In his solution for the jet in an infinite stream Pack observes that there is a singular case when $\beta_1\beta_2 = 1$ (that is when $m = 1$); the jet boundary expands to a width given by $f(x) = \epsilon$ and thereafter remains at this constant width. When the outer stream is of finite width and $m = 1$, expansion of (39) leads to the result

$$\bar{F}(\xi) = [1-(a+b)+ab][1+ab+a^2b^2+a^3b^3+\ldots]/2i\xi^2.$$

where $a = \exp(-2i\xi)$ and $b = \exp(-2ik\xi)$. The transition to an infinite stream is made by letting k tend to infinity. If $k = 1$ then

$$\bar{F}(\xi) = [1-2\exp(-2i\xi)\{1-\exp(-2i\xi)+\exp(-4i\xi)+\exp(-6i\xi)+\ldots\}]/2i\xi^2$$

and this leads to the term-by-term transform

$$F(t) = \tfrac{1}{2}[\{t\}-2\{t-2\}+2\{t-4\}-2\{t-6\}+\ldots],$$

where $$\{t-T\} = \begin{cases}(t-T) & \text{if } t \geqslant T,\\ 0 & \text{if } t < T.\end{cases}$$

The graph of this function is a well-known triangular type waveform and is illustrated in figure 2. The boundary expands to $F(2) = 1$ and then oscillates as shown between $F(t) = 1$ and $F(t) = 0$. If $k = 2$, then

$$\bar{F}(s) = [1-a-a^2+2a^3-a^4-a^5+2a^6\ldots]/2i\xi^2$$

and so $$F(t) = \tfrac{1}{2}[\{t\}-\{t-2\}-\{t-4\}+2\{t-6\}\ldots].$$

The boundary expands to $F(2) = 1$, remains at constant width in the interval $2 \leqslant t \leqslant 4$ and then contracts to $F(6) = 0$. This waveform is then repeated as t increases. The waveform when $k = 10$ is shown in figure 3. As k increases the disturbance due to the arrival at the boundary of the wave reflected from the outer stream wall moves further downstream, and it disappears as k becomes infinitely large. The jet boundary then expands to $F(2) = 1$ and remains constant.

It may be observed that for $m = 1$, the oscillations in the jet boundary are periodic with a period in t of $2(k+1)$ when k is rational. When k is irrational, the waveform is of a similar shape but no longer periodic.

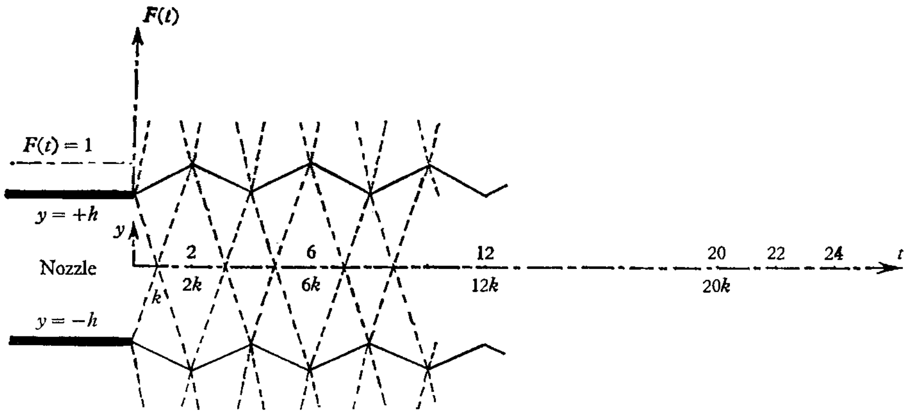

FIGURE 2. Oscillation of the jet boundary. Case of no reflected waves at boundary; $m = 1, k = 1$. (The Mach lines are shown dotted. The outer rigid walls are not shown.)

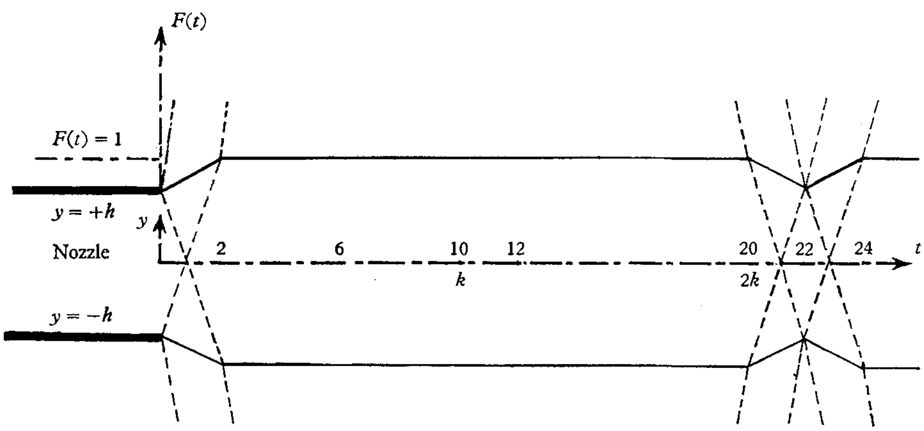

FIGURE 3. Oscillation of the jet boundary. Case $m = 1, k = 10$.

A further investigation into the nature of the reflected and transmitted waves may be made by expanding the inverse transforms of the perturbation potentials $\psi(t, y)$ in each of the two regions in powers of exponential functions and then transforming term by term (Lennox 1959). The wave disturbance inside the jet is complicated by the presence of disturbances which are transmitted through the boundary and then reflected back again at the outer walls. The transmission and reflexion coefficients may be found and the work of Pack (1956) extended.

5. The subsonic jet in a subsonic stream

When both the jet and outer stream are subsonic a formulation of the problem in the manner of the preceding sections leads again to a solution possessing an infinite gradient at the orifice. The width of the jet changes rapidly from the orifice towards its value at infinity downstream given by

$$y = h[1 - \epsilon\beta_1\beta_2 k/(mk+1)].$$

In the expressions for the potentials there are no periodic or almost periodic terms, the terms in the summations decaying exponentially with distance both upstream and down from the jet orifice.

Here again it might be possible to eliminate the anomalous gradient at the jet orifice by designing the nozzle suitably.

6. Conclusions

The problem of a two-dimensional compound jet has been solved by using the Wiener–Hopf technique with the linearized theory. The solutions have been presented in the form of infinite series. Downstream of the jet exit (or of the leading characteristics from the end of the nozzle in the supersonic motion) the perturbation potential consists essentially of two parts; the first part contains attenuation terms which decay exponentially with the distance from the exit and the second part contains terms which do not decay but are of quasi-periodic nature. Upstream of the jet exit the perturbation potential is zero when the flow is supersonic and contains only attenuation terms when the flow is subsonic.

It is found that the boundary of the jet fluctuates about a mean displacement, measured from the undisturbed position of the boundary, whose magnitude is given by $\epsilon h\beta_1\beta_2 k/(mk \mp 1)$, the plus sign to be taken when both flows are supersonic. When only one of the flows is supersonic the nature of the solution, and in particular the ultimate mean width and the initial slope of the boundary, is found to depend on the magnitude of mk, a quantity depending upon the gas constants, the Mach numbers and stream widths, relative to unity. In the over-pressure case ($\epsilon > 0$) the supersonic jet expands into the subsonic region when $mk > 1$, whereas the subsonic jet expands into the supersonic stream when $mk < 1$. When the inequalities for mk are reversed the solution contains an infinite gradient in the boundary of the jet at the orifice; the physical significance of this has been considered. The case $mk = 1$ is critical, the perturbation potentials and the boundary displacement becoming infinite with the distance downstream from the jet orifice. The solution then has the appearance of a resonance effect and would seem to indicate a breakdown in the applicability of the linearized theory.

It is shown that the asymptotic form of the solution to the problem of a supersonic jet in a subsonic stream provides the solution originally presented by Pai (1952). The results obtained by Pack (1956) for the flow of a supersonic jet in a supersonic stream are confirmed and extended.

Editor's note: A point in dispute between the authors and a referee has been left unresolved. The referee's position is as follows:

It is the opinion of the referee that in the cases of §§ 3.2 and 5 there is an essential non-uniqueness in the physical solutions, one which does not appear in the author's solutions because they do not provide for upstream propagation of disturbances in the upstream subsonic channels. The downstream flow in the case of § 3 is mixed subsonic-supersonic, acts like a subsonic flow in permitting upstream propagation of a disturbance if $mk > 1$, and acts like a supersonic flow in forbidding upstream propagation of a disturbance if $mk < 1$. For each subsonic channel in the upstream flow there is a parameter to be determined which measures the strength of the upstream disturbance in that channel. In the case of § 3.1 ($mk > 1$) the pressure downstream determines the single parameter in question. In the case of § 3.2 the parameter is indeterminate. In the case of § 5 the pressure downstream can determine at most one relation connecting the two parameters in question, so that one parameter remains indeterminate.

These two indeterminate cases show the phenomenon of an infinite *negative* boundary slope near the orifice in the solutions of this paper. The referee feels that this phenomenon is anomalous. This fact suggests that the proper resolution of the indeterminancy is through a Kutta condition at this point. With a Kutta condition applied, the solution is one with each stream at constant velocity, and is then unique.

REFERENCES

CARSLAW, H. S. & JAEGER, J. C. 1959 *Conduction of Heat in Solids*, 2nd ed. Oxford University Press.
CHAPLYGIN, A. 1904 *Ann. Sci. Univ. Moscow* **21**, 1; translated in *N.A.C.A., T.M.* 1063 (1944).
EHLERS, F. E. & STRAND, T. 1958 *J. Aero/Sp. Sci.* **25**, 497.
EMDEN, R. 1899 *Ann. Phys., Lpz.*, **69**, 264, 426.
HARDY, G. H. 1938 *Pure Mathematics* (7th ed.). Cambridge University Press.
JACOB, C. 1936 *C.R. Acad. Sci., Paris*, **203**, 423.
KAWAMURA, R. 1952 *J. Phys. Soc. Japan*, **7**, 482.
KLUNKER, E. B. & HARDER, I. C. 1952 *J. Aero/Sp. Sci.* **19**, 427.
LENNOX, S. C. 1959 Ph.D. Thesis, Edinburgh University.
MACH, E. & SALCHER, P. 1890 *Ann. Phys., Lpz.*, **41**, 144.
NOBLE, B. 1958 *The Wiener–Hopf Technique*. London: Pergamon.
PACK, D. C. 1948 *Quart. J. Mech. Appl. Math.* **1**, 1.
PACK, D. C. 1950 *Quart. J. Mech. Appl. Math.* **3**, 176.
PACK, D. C. 1956 *J. Aero/Sp. Sci.* **23**, 747.
PAI, S. I. 1952 *J. Aero/Sp. Sci.* **19**, 61.
PALEY, R. E. A. C. & WIENER, N. 1934 *Amer. Math. Soc. Colloquia Publ. New York*, no. 19.
PRANDTL, L. 1904 *Phys. Z.* **5**, 599.
PRANDTL, L. 1907 *Phys. Z.* **8**, 23.
RAYLEIGH, LORD 1916 *Phil. Mag.* **32**, 177.
STOKER, J. J. 1957 *Water Waves*. New York: Interscience.
WOODS, L. C. 1961 *The Theory of Subsonic Plane Flow*. Cambridge University Press.